

# Tomography of Quantum Operations

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(February 1, 2008)

Quantum operations describe any state change allowed in quantum mechanics, including the evolution of an open system or the state change due to a measurement. In this letter we present a general method based on quantum tomography for measuring experimentally the matrix elements of an arbitrary quantum operation. As input the method needs only a single entangled state. The feasibility of the technique for the electromagnetic field is shown, and the experimental setup is illustrated based on homodyne tomography of a twin-beam.

The typical state change in quantum mechanics is the unitary evolution, where the final state is related to the initial one via the transformation  $\rho \rightarrow \mathcal{E}(\rho) \equiv U\rho U^\dagger$ , with  $U$  unitary operator on the Hilbert space  $\mathcal{H}$  of the system. Unitary transformations describe only the evolutions of closed systems, and non-unitary transformations occur when the quantum system is coupled to an environment or when a measurement is performed on the system. What is the most general possible state change in quantum mechanics? The answer is provided by the formalism of “quantum operations” by Kraus [1]. Here *input* and *output* states are connected via the map

$$\rho \rightarrow \frac{\mathcal{E}(\rho)}{\text{Tr}(\mathcal{E}(\rho))}. \quad (1)$$

The *quantum operation*  $\mathcal{E}$  is a linear, trace-decreasing map that preserves positivity (more precisely the map must be completely positive (CP) [2]). The trace in the denominator is included in order to preserve the normalization  $\text{Tr}(\rho) = 1$ . The most general form for  $\mathcal{E}$  can be shown to be [1]

$$\mathcal{E}(\rho) = \sum_n K_n \rho K_n^\dagger, \quad (2)$$

where the operators  $K_n$  satisfy the bound

$$\sum_n K_n^\dagger K_n \leq I. \quad (3)$$

The transformation (2) occurs with generally non-unit probability  $\text{Tr}(\mathcal{E}(\rho)) \leq 1$ , and the probability is unit only when  $\mathcal{E}$  is trace-preserving, i.e. when the bound (3) is achieved with the equal sign. The particular case of unitary transformations corresponds to having only one term  $K_1 = U$  in the sum (2), with  $U$  unitary. However,

one can consider also non-unitary operations with only one term, i. e.

$$\mathcal{E}(\rho) = A\rho A^\dagger, \quad (4)$$

with  $A$  a *contraction*, i. e.  $\|A\| \leq 1$ : we'll call these last operations *pure*, since they leave pure states  $\rho$  as pure. Indeed, for  $\rho = |\varphi\rangle\langle\varphi|$  we can rewrite Eq. (1) in the form

$$|\varphi\rangle \rightarrow \frac{A|\varphi\rangle}{\|A|\varphi\rangle\|} \quad (5)$$

Such an operation could, for example, describe the state reduction from a measurement apparatus for a given fixed outcome, which occurs with probability  $\text{Tr}(\rho A^\dagger A) \leq 1$ .

Suppose now that we have a quantum machine that performs an unknown quantum operation  $\mathcal{E}$ , and we want to determine  $\mathcal{E}$  experimentally. This problem has been posed in several papers, with solutions given in some special cases [3–5].

How can we do? This would be the case, for example, if we want to determine the unitary transformation  $U$  performed by a quantum device, or the state-reduction achieved by a measuring apparatus that performs an indirect measurement on the system. In Refs. [6,7] as a method it was suggested to carry on a tomographic reconstruction at the machine output for a varying input state. However, the availability of all possible input states is a practically unsolvable problem. For example, the method of Ref. [7] in the optical domain works only for phase-insensitive devices, since for phase-sensitive ones one would need input superpositions of two photon-number states, superpositions which are currently not feasible. As we will show in this letter, we can exploit the *quantum parallelism* of entanglement [8] to run all possible input states in parallel using only a single entangled state as the input in the tomographic reconstruction. In this way we have at our disposal a general method for experimentally determining the quantum operation matrix, using any available quantum-tomographic scheme for the system in consideration, and a single fixed state at the input, which is an entangled (not even maximally) state. In the optical domain we will show that one can achieve the tomographic reconstruction of the operation using exactly the same apparatus of the recently performed experiment of Ref. [9].

Let's consider for simplicity a “pure” quantum operation in the form (5). Given an orthonormal basis  $\{|j\rangle\}$  corresponding to some physical observable, how can we determine the matrix  $A_{ij} = \langle i|A|j\rangle$  experimentally? Instead of acting with the contraction  $A$  on a “isolated”

system, we perform the map on a system which is entangled in the state  $|\psi\rangle\langle\psi| \in \mathcal{H} \otimes \mathcal{H}$  with an identical system, i. e.

$$|\psi\rangle\langle\psi| \rightarrow |\phi\rangle\langle\phi| = \frac{A \otimes I|\psi\rangle\langle\psi|}{\|A\psi\|_{HS}}. \quad (6)$$

With the double ket we denote bipartite vectors  $|\psi\rangle\langle\psi| \in \mathcal{H} \otimes \mathcal{H}$ , which, keeping the basis  $\{|j\rangle\}$  as fixed, are in one-to-one correspondence with matrices as follows

$$|\psi\rangle\langle\psi| = \sum_{ij} \psi_{ij} |i\rangle\langle j|. \quad (7)$$

In the following we will also use the simple notation of using the same symbol  $A$  for both the matrix  $A = \{A_{ij}\}$  and the corresponding operator  $A = \sum_{ij} A_{ij} |i\rangle\langle j|$  for fixed basis  $\{|j\rangle\}$ . With this notation the norm  $\|A\|_{HS}$  in Eq. (6) denotes the Hilbert-Schmidt norm  $\|A\|_{HS} = [\text{Tr}(A^\dagger A)]^{\frac{1}{2}}$ . We'll also denote by  $A^*$  the operator corresponding to the complex conjugated matrix of  $A$  (with respect to the same fixed basis  $\{|j\rangle\}$ ), and analogously  $A^T$  will denote the transposed-matrix operator. With consistent notation we'll write  $A = \{A_{ij}\} \equiv [A(j)]$  to denote the column vectors  $A(j)$  of the matrix  $A$ , and use  $|A(j)\rangle = A|j\rangle \equiv \sum_i A_{ij} |i\rangle$  for the corresponding vectors in  $\mathcal{H}$ . Using this simple formalism, the quantum operation matrix  $A$  in terms of the input and output state-matrices writes as follows

$$A = \phi \psi^{-1} \sqrt{p_A(\psi)}, \quad (8)$$

where  $p_A(\psi) = \|A\psi\|_{HS}^2$  denotes the occurrence probability of the quantum operation, and the entangled state is assumed to have invertible matrix  $\psi$  (which is always the case in practice). In our matrix formalism the matrix  $\phi$  corresponding to the output state can be written in terms of measurable ensemble averages as follows

$$\phi_{ij} \equiv \langle\langle i, j | \phi \rangle\rangle = e^{i\theta} \frac{\langle |i_0, j_0\rangle\langle i, j| \rangle}{\sqrt{\langle |i_0, j_0\rangle\langle i_0, j_0| \rangle}}, \quad (9)$$

where  $\langle\dots\rangle \equiv \langle\langle \phi | \dots | \phi \rangle\rangle$  denotes the ensemble at the output,  $|i, j\rangle \equiv |i\rangle \otimes |j\rangle$ ,  $i_0, j_0$  are suitable fixed integers, and  $e^{i\theta}$  is an irrelevant (unmeasurable) overall phase factor corresponding to  $\theta = \arg(\langle |i_0, j_0 | \phi \rangle)$ . Using Eq. (8) we can write the matrix  $A_{ij}$  in terms of only output ensemble averages as follows

$$A_{ij} = \kappa \langle E_{ij}(\psi) \rangle, \quad (10)$$

where the operator  $E_{ij}(\psi)$  is given by

$$E_{ij}(\psi) = |i_0\rangle\langle i| \otimes |j_0\rangle\langle \psi^{-1*}(j)|, \quad (11)$$

and the proportionality constant is given by

$$\kappa = e^{i\theta} \sqrt{\frac{p_A(\psi)}{\langle |i_0, j_0\rangle\langle i_0, j_0| \rangle}}. \quad (12)$$

Since  $A_{ij}$  is written only in terms of output ensemble averages, it can be estimated through quantum tomography. Quantum tomography [10] is a method to estimate the ensemble average  $\langle H \rangle$  of any arbitrary operator  $H$  on  $\mathcal{H}$  by using only measurement outcomes of a *quorum* of observables  $\{O(l)\}$ . A *quorum* is just a set of operators  $\{O(l)\}$  which are observable (i.e. have orthonormal resolution), and span the linear space of operators on  $\mathcal{H}$ . This means that any operator  $H$  can be expanded as  $H = \sum_l \text{Tr}[Q^\dagger(l)H]O(l)$ , where  $\{Q(l)\}$  and  $\{O(l)\}$  form a biorthogonal set such that  $\text{Tr}[Q^\dagger(i)O(j)] = \delta_{ij}$ . Hence, the tomographic estimation of the ensemble average  $\langle H \rangle$  is obtained as the double average—over both the ensemble and the quorum—of the unbiased estimator  $\text{Tr}[Q^\dagger(l)H]O(l)$  with random  $l$ . The most popular example of quantum tomography is homodyne tomography [11], where the quorum (selfdual) is given by the operators  $\exp(ikX_\phi)$  for varying  $k$  and  $\phi$ ,  $X_\phi$  denoting a quadrature of one mode of radiation. Notice that for estimating the density matrix also the maximum-likelihood strategy can be used instead of averaging [12,13]. Moreover, there is a general method [13] for deconvolving instrumental noise when measuring the quorum, which resorts to finding the biorthogonal basis for the noisy quorum. This is the case, for example, of deconvolution of noise from non-unit quantum efficiency in homodyne tomography [11]. Finally, for multipartite quantum systems, one can simply use as a quorum the tensor product of single-system quorums [13]: this means that, in our case, we just need to make two local quorum measurements jointly on the two systems, and analyze data with the tensor-product estimators. For example, the estimation of  $A_{ij}$  in Eq. (10) resorts to the calculation of the following ensemble average from the experimental data

$$A_{ij} = \left\langle \kappa \sum_{kl} a_{ij}(kl) O(k) \otimes O(l) \right\rangle, \quad (13)$$

where the c-numbers  $a_{ij}(kl)$  are given by

$$a_{ij}(kl) = \langle i | Q^\dagger(k) | i_0 \rangle \langle \psi^{-1*}(j) | Q^\dagger(l) | j_0 \rangle. \quad (14)$$

Also the fixed ensemble average  $\langle |i_0, j_0\rangle\langle i_0, j_0| \rangle$  in the constant  $\kappa$  can be measured via tomography, or even by coincidence counting, whereas  $p_A(\psi)$  results from counting the occurrence of  $A$  (occurrence is checked by reading the apparatus, e. g.  $A$  is in correspondence with a given measurement outcome).

The general experimental scheme of the method for the tomographic estimation of a quantum operation matrix is sketched in Fig. 1.

The method given above can be easily generalized to the case of arbitrary non pure quantum operation, as in Eqs. (1) and (2). Now the output state is the joint density matrix

$$\begin{aligned} |\psi\rangle\langle\psi| \rightarrow R(\psi) &\equiv \mathcal{E} \otimes \mathcal{I}(|\psi\rangle\langle\psi|) \\ &\equiv \sum_n K_n \otimes I |\psi\rangle\langle\psi| K_n^\dagger \otimes I. \end{aligned} \quad (15)$$

One can immediately see that the quantum operation can be written in terms of the density matrix  $R(\psi)$  for  $\psi = I$ , i. e.

$$\mathcal{E}(\rho) = \text{Tr}_2[I \otimes \rho^T R(I)], \quad (16)$$

where  $\text{Tr}_2$  denotes the partial trace on the second Hilbert space. However, for invertible  $\psi$  the two matrices  $R(I)$  and  $R(\psi)$  are connected as follows

$$R(I) = (I \otimes \psi^{-1T})R(\psi)(I \otimes \psi^{-1*}). \quad (17)$$

Hence, the (four-index) matrix  $R$  in Eq. (16) which is in one-to-one correspondence with the quantum operation  $\mathcal{E}$  can be obtained by estimating via quantum tomography the following output ensemble averages

$$\begin{aligned} \langle\langle i, j | R(I) | l, k \rangle\rangle &= \langle E_{lk}^\dagger(\psi) E_{ij}(\psi) \rangle = \\ &= \langle |l\rangle\langle i| \otimes |\psi^{-1*}(k)\rangle\langle\psi^{-1*}(j)| \rangle. \end{aligned} \quad (18)$$

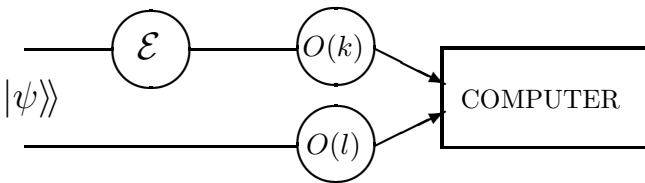


FIG. 1. General experimental scheme of the method for the tomographic estimation of a quantum operation. Two identical quantum systems are prepared in an entangled state  $|\psi\rangle\langle\psi|$ . One of the two systems undergoes the quantum operation  $\mathcal{E}$ , whereas the other is left untouched. At the output one makes a quantum tomographic estimation, photocurrent by measuring jointly two random observables from a quorum  $\{O(l)\}$  (see the text).

We now analyze the experimental feasibility of the method in the optical domain, based on tomographic homodyning a twin-beam from parametric downconversion of the vacuum. As a simple example of quantum operation we consider the unitary displacement  $D(z) = \exp(z a^\dagger - z^* a)$ , of a single radiation mode with annihilation and creation operators  $a$  and  $a^\dagger$ . The experimental apparatus is the same as in the experiment of Ref. [9], with a nondegenerate optical parametric amplifier (a KTP crystal) pumped by the second harmonic of a Q-switched mode-locked Nd:YAG laser, which produces a 100-MHz train of 120-ps duration pulses at 1064 nm. The orthogonally polarized twin beams emitted by the KTP crystal (one of which is displaced of  $D(z)$  by a nearly transparent beam splitter with a strong local oscillator) are separately detected by two balanced homodyne setups that use two independent local oscillators derived from the same laser, with the amplified output noise at radio-frequencies downconverted to the near-dc by use of an rf mixer and sampled by a boxcar integrator. The outputs of the boxcar channels are a measure of the quadrature amplitudes  $X_{\phi'} \otimes X_{\phi''}$  for random phases  $\phi'$  and  $\phi''$

with respect to the local oscillators, where the quadratures  $X_\phi = \frac{1}{2}(a^\dagger e^{i\phi} + a e^{-i\phi})$  here represent the quorum of observables for the tomographic reconstruction (for additional details on the experimental setup see Ref. [9], whereas for a more extensive theoretical treatment see Ref. [14]).

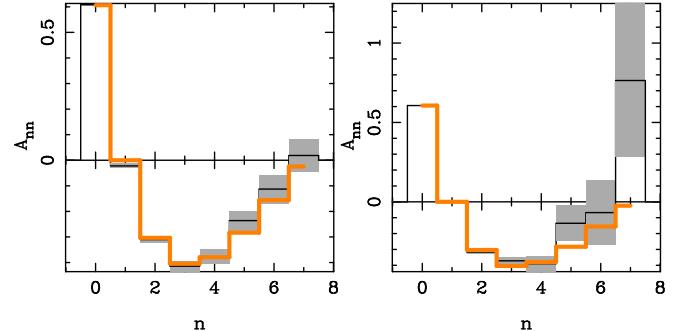


FIG. 2. Homodyne tomography of the quantum operation  $A$  corresponding to the unitary displacement of one mode of the radiation field. Diagonal elements  $A_{nn}$  (shown by thin solid line on an extended abscissa range,) with their respective error bars in gray shade, compared to the theoretical probability (thick solid line). Similar results are obtained for all upper and lower diagonals of the quantum operation matrix  $A$ . The reconstruction has been achieved using an entangled state  $|\psi\rangle\langle\psi|$  at the input corresponding to parametric downconversion of vacuum with mean thermal photon  $\bar{n}$  and quantum efficiency at homodyne detectors  $\eta$ . Top:  $z = 1$ ,  $\bar{n} = 5$ ,  $\eta = 0.9$ , and 150 blocks of  $10^4$  data have been used. Bottom:  $z = 1$ ,  $\bar{n} = 3$ ,  $\eta = 0.7$ , and 300 blocks of  $2 \cdot 10^5$  data have been used. The bottom plot corresponds to the same parameters of the experiment in Ref. [9].

In Fig. 2 the results from a homodyne tomography of an optical displacement of one of the two twin beams from parametric downconversion of the vacuum are presented for a simulated experiment, for displacement parameter  $z = 1$ , and for some typical values of the quantum efficiency  $\eta$  at homodyne detectors and of the mean thermal photon number  $\bar{n}$  of the twin beam. As one can see a meaningful reconstruction of the matrix can be achieved in the given range with  $10^6 \div 10^7$  data, but this number can be decreased of a factor  $100 - 1000$  using the tomographic max-likelihood techniques of Ref. [12], however at the expense of the complexity of the algorithm. Homodyne overall quantum efficiencies and amplifier gains (for the twin-beam) typical of the experimental setup of Ref. [9] are considered. Improving quantum efficiency and increasing the amplifier gain (toward a maximally entangled state) have the effect of making statistical errors smaller and more uniform versus the photon labels  $n$  and  $m$  of the matrix  $A_{nm}$ . Meaningful reconstructions can be achieved with as few as  $\bar{n} \sim 1$  thermal photons, and with quantum efficiency as low as  $\eta = 0.7$ .

We want to mention that the present quantum tomographic method for measuring the matrix of a quantum

operation can be much improved by means of a max-likelihood strategy aimed at the estimation of some unknown parameters of the quantum operation (such max-likelihood strategy should not be confused with the max-likelihood method of for the tomographic reconstruction in Ref. [12]). In this case, instead of obtaining the matrix elements of  $R(I)$  from the ensemble averages in (18), one has  $R(I)$  parametrized in terms of unknown quantities to be experimentally determined, and the likelihood is maximized for the set of experimental data at various randomly selected (tensor) quorum elements, keeping the same fixed entangled input state. This method is especially useful for a very precise experimental comparisons between the characteristics of a given device (e.g. the gain and loss of an active fiber) with those of a *quantum standard* reference [15].

In conclusion, in this letter we have presented a general tomographic method for measuring the matrix of any quantum operation of arbitrary quantum system. The method exploits the quantum parallelism of entanglement, with a single entangled state playing the role of a varying input state, thus overcoming the practically unsolvable problem of availability of all possible input states for the tomographic analysis of the quantum operation. We have shown the feasibility of the method for the case of the electromagnetic field via homodyne tomography of a twin-beam from nondegenerate downconversion of the vacuum. The unilateral displacement of the twin-beam has been considered, and for displacement parameters of the order of unit our results show that the tomographic estimation can be achieved using the same apparatus of a similar recently performed experiment.

This work has been supported by the Italian Ministero dell'Università e della Ricerca Scientifica e Tecnologica (MURST) under the co-sponsored project 1999 *Quantum Information Transmission And Processing: Quantum Teleportation And Error Correction*.

- (1997)
- [7] G. M. D'Ariano and L. Maccone, Phys. Rev. Lett. **80** 5465 (1998)
- [8] *Introduction to Quantum Computation and Information*, Ed. by H.-K. Lo, S. Popescu, T. Spiller, World Scientific (Singapore 1998).
- [9] M. Vasilyev, S.-K. Choi, P. Kumar, and G. M. D'Ariano, Phys. Rev. Lett. **84** 2354 (2000)
- [10] For a recent paper on quantum tomography presenting the general approach of the present paper, along with examples, see: G. M. D'Ariano, L. Maccone, and M. G. A. Paris, Phys. Lett. A **276** 25 (2000). For a starting point on the extensive literature on this topic see references therein.
- [11] For a review, see: G. M. D'Ariano, *Measuring Quantum States*, in *Quantum Optics and Spectroscopy of Solids*, ed. by T. Hakioglu and A. S. Shumovsky, (Kluwer Academic Publisher, Amsterdam 1997), p. 175-202
- [12] K. Banaszek, G. M. D'Ariano, M. G. A. Paris, M. Sacchi, *Maximum-likelihood estimation of the density matrix*, Phys. Rev A **61**, 010304 (2000) (rapid communication)
- [13] G. M. D'Ariano, *Universal quantum estimation*, Phys. Lett. A **268** 151 (2000)
- [14] G. M. D'Ariano, M. Vasilyev, and Prem Kumar *Self-homodyne tomography*, Phys. Rev. A **58** 636 (1998)
- [15] G. M. D'Ariano, P. Lo Presti, and M. G. A. Paris, unpublished.

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- [1] K. Kraus, *States, Effects, and Operations* (Springer-Verlag, Berlin, 1983).
- [2] A map  $\mathcal{E}$  is completely positive if it preserves positivity not just for a local state in  $\mathcal{H}$ , but also for any state of the system that is entangled with any other system. In other words, upon denoting by  $\mathcal{I}$  the identical map on the Hilbert space  $\mathcal{K}$  of a second quantum system, the extended map  $\mathcal{E} \otimes \mathcal{I}$  on  $\mathcal{H} \otimes \mathcal{K}$  must be positive for any extension  $\mathcal{K}$ .
- [3] Q. A. Turchette, C. J. Hood, W. Lange, H. Mabuchi, and H. J. Kimble, Phys. Rev. Lett. **75** 4710 (1995)
- [4] J. F. Poyatos, J. I. Cirac, and P. Zoller Phys. Rev. Lett. **78** 390 (1997)
- [5] K. R. W. Jones, Phys. Rev. A **50** 3682 (1994)
- [6] I. L. Chuang and M. A. Nielsen, J. Mod. Opt. **44** 2455